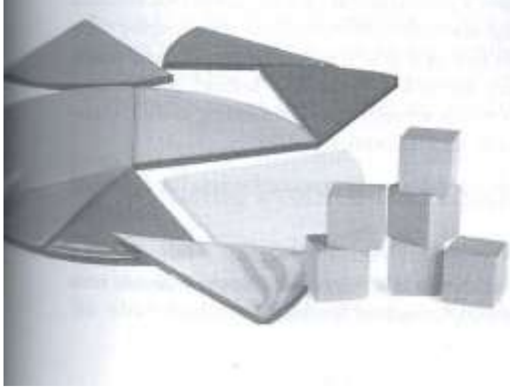
2

Teaching Mathematics through Problem Solving

As it turns out, understanding is supported best through a delicate balance among engaging students in solving challenging problems, examining increasingly better solution methods, and providing information for students at just the right times.

Hiebert and Wearne (2003, p. 5)*

Teaching mathematics *through* problem solving is a method of teaching mathematics that helps students develop relational understanding. With this approach, problem solving is completely interwoven with learning. As students *do* mathematics—make sense of cognitively demanding tasks, provide evidence or justification for strategies and solutions, find examples and connections, and receive and provide feedback about ideas—they are simultaneously engaged in the activities of problem solving and learning. Teaching mathematics through problem solving requires you to think about the types of tasks you pose to students, how you facilitate discourse in your classroom, and how you support students' use of a variety of representations as tools for problem solving, reasoning, and communication.



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Teaching Student-Centered Mathematics – Developmentally Appropriate Instruction for Grades 6-8
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Teaching through Problem Solving: An Upside-Down Approach

For many years and continuing today, mathematics has been taught using a teaching-*for*-problem-solving approach: The teacher presents the mathematics, the students practice the skill, and finally, the students solve word problems that require using that skill. Unfortunately, this "do-as-I-show-you" approach to mathematics teaching has not been successful for helping many students understand or remember mathematics concepts (e.g., Pesek & Kirshner, 2002; Philipp & Vincent, 2003).

Teaching mathematics *through* problem solving generally means that students solve problems to learn new mathematics rather than just apply mathematics after it has been learned. Students learn mathematics through real contexts, problems, situations, and models that allow them to build meaning for the concepts (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997). So teaching *through* problem solving might be described as "upside-down" from the traditional approach of teaching *for* problem solving because the problem is presented at the beginning of a lesson and skills and ideas emerge from working with the problem. An example of teaching through problem solving might have students explore the following situation before they are taught how to set up proportions and solve for the unknown.

Tatyana has a coupon for 4 pizzas for $10. If the restaurant will give her the same rate for multiple pizzas, how much will 18 pizzas cost?

The teacher would explain to the class that there is more than one correct strategy to solve this problem and that they are to find as many different strategies as they can. Students might use counters, create a drawing, make an organized list, or solve through a series of operations.

Stop and Reflect

Find a way to determine how much money 18 pizzas will cost without using the standard algorithm of setting up proportions and solving for the unknown. ■

Through this context and exploration, students could see how to use ratio tables, which can be used to solve other real-world problems involving proportions. The pizza problem generates opportunities for students to improve their multiplicative reasoning as they find ways to relate numbers whose multiplicative relationship is not readily apparent. Using the standard algorithm of setting up proportions and solving for the unknown can take away the opportunity for students to develop multiplicative and proportional reasoning.

Teaching *through* problem solving requires a paradigm shift, which means that teachers are doing more than just tweaking a few things about their teaching; they are changing their philosophy of how they think students learn best and how they can best help them learn. At first glance, it may seem that the teacher's role is less demanding because the students are doing the mathematics, but the teacher's role is actually more demanding in such classrooms. Here are some of the important teacher responsibilities:

- Select high-quality tasks that allow students to learn the content by figuring out their own strategies and solutions.
- Ask high-quality questions that allow students to verify and relate their strategies.
- Listen to students' responses and examine their work, determining in the moment how to extend and formalize their thinking through targeted feedback.

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There is no doubt that teaching mathematics through problem solving can be challenging, but the results are worth the effort! It promises to be a better approach if our ultimate goal is deep (relational) understanding because teaching through problem solving accomplishes these goals:

- *Focuses students' attention on ideas and sense making.* When solving problems, students are necessarily reflecting on the concepts inherent in the problems. Emerging concepts are more likely to be integrated with existing ones, thereby improving understanding.
- *Emphasizes mathematical processes and practices.* Students who are solving problems will engage in all five of the processes of doing mathematics—problem solving, reasoning, communication, connections, and representation (NCTM, 2000), as well as the eight mathematical practices outlined in the *Common Core State Standards*, resulting in mathematics that is more accessible, more interesting, and more meaningful. Note that the first Standard for Mathematical Practice is "Make sense of problems and persevere in solving them" (CCSSO, 2010).
- *Develops students' confidence and identities.* Every time teachers pose a problem-based task and expect a solution, they implicitly say to students, "I believe you can do this." When students are engaged in problem solving and discourse in which the correctness of the solution lies in the justification of the process, they begin to see themselves as capable of doing mathematics and that mathematics makes sense.
- *Provides a context to help students build meaning for the concept.* Using a context facilitates mathematical understanding, especially when the context is grounded in an experience familiar to students and when the context uses purposeful constraints that potentially highlight the significant mathematical ideas (Fosnot & Dolk, 2001).
- *Allows entry and exit points for a wide range of students.* Good problem-based tasks have multiple paths to the solution, so each student can make sense of and solve the task by using his or her own ideas. Furthermore, students expand their ideas and grow in their understanding as they hear, critique, and reflect on the solution strategies of others.
- *Allows for extensions and elaborations.* Extensions and "what-if" questions can motivate advanced learners or quick finishers, resulting in increased learning and enthusiasm for doing mathematics.
- *Engages students so that there are fewer discipline problems.* Many discipline issues in a classroom are the result of students becoming bored, not understanding the teacher directions, or simply finding little relevance in the task. Most students like to be challenged and enjoy being permitted to solve problems in ways that make sense to them, giving them less reason to act out or cause trouble.
- *Provides formative assessment data.* As students discuss ideas, draw diagrams, or use manipulatives, defend their solutions and evaluate those of others, and write reports or explanations, they provide the teacher with a steady stream of valuable information that can be used to inform subsequent instruction.
- *Is a lot of fun!* Students enjoy the creative process of problem solving and sharing how they figured something out. After seeing the surprising and inventive ways that students think and how engaged students become in mathematics, very few teachers stop using a teaching-*through*-problem-solving approach.

Using Problems to Teach

When teachers teach mathematics through problem solving, students learn the desired content through problems (tasks or activities). A *problem* is defined here as any task or activity for which students have no prescribed or memorized rules or methods, and for which they

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do not have a perception that there is a specific "correct" solution method (Hiebert et al., 1997). In other words, the task or activity is a genuine problem.

Features of a Problem

Problems that can serve as effective tasks or activities for students to solve have common features. Use the following points as a guide to assess whether a task or an activity has the potential to be a genuine problem.

- *The problem should engage students where they are in their current understanding.* Students should have the appropriate ideas to begin engaging with the problem and to solve the problem, yet still find it challenging and interesting.
- *The problematic or engaging aspect of the problem must be a result of the mathematics that the students are to learn.* In solving the problem or doing the activity, students should be concerned primarily with making sense of and developing their understanding of the mathematics involved. Any context or motivation used should not overshadow the mathematics to be learned.
- *The problem must require justifications and explanations for answers and methods.* In a high-quality problem, neither the process nor the answer is straightforward, so justification is central to the task. Students should understand that the responsibility for determining whether answers are correct and why they are correct rests on their mathematical reasoning, not on the teacher telling them that they are correct.

Examples of Problems

Problems can be used to develop both concepts and procedures, and the connection between concepts and procedures. In the following examples, the first two problems focus on concepts, and the third problem focuses on a procedure.

CONCEPT: Comparing Ratios and Proportional Reasoning

Jack and Jill were at the same spot at the bottom of a hill, hoping to fetch a pail of water. They both begin walking up the hill, Jack walking 5 yards every 25 seconds and Jill walking 3 yards every 10 seconds. Assuming a constant walking rate, who will get to the pail of water first?

CONCEPT: Equality

$$64 \div 16 = 32 \div b$$

Find a number for b so that the equation is true. Is there more than one number that will make the equation true? Why or why not? Can you find more than one way to find a number for b so that the equation is true?

Note that a task in the form of a story problem does not automatically make the task a problem. A story problem can be "routine" if students read it and know right away that it is a division problem and divide to answer it. Conversely, an equation with no words, as in the second example above, is not necessarily routine and can actually be a rich problem to investigate.

PROCEDURE: Dividing Two Fractions

Solve this problem in two different ways: $\frac{3}{4} \div \frac{1}{2} =$ ____________.

For each way, explain how you solved it.

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The third example, although focused on a procedure, is a problem because students must figure out *how* they are going to approach the task (assuming they have not been taught the standard algorithm at this point). Students are also challenged to find more than one way to solve the problem. Implicit is the challenge to determine how the two solution strategies are different. The third example is important because it illustrates that virtually all mathematics—concepts and procedures—can be taught through problem solving.

Selecting Worthwhile Tasks

As noted earlier in the three features of a problem, a task must engage students where they currently are in their understanding and simultaneously must be problematic for the students. In selecting such a task, consider the level of cognitive demand, the potential of the task to have multiple entry and exit points, and the relevancy of the task to students.

Level of Cognitive Demand

Research supports the practice of engaging students in productive struggle to develop understanding (Bay-Williams, 2010; Hiebert & Grouws, 2007). Both words in the phrase "productive struggle" are important. Students must have the tools and prior knowledge to solve a problem and not be given a problem that is out of reach because otherwise they will struggle without being productive; however, students should not be given tasks that are straightforward and trivial because they will not struggle with mathematical ideas and further develop their understanding. When students know that struggle is an expected part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008).

Figure 2.1 shows a useful framework for determining whether a task has the potential to challenge students (Smith & Stein, 1998). The framework distinguishes between tasks that require low levels and high levels of cognitive demand. Tasks that have low-level cognitive demand are routine and straightforward and do not engage students in productive struggle. Tasks with a high level of cognitive demand not only engage students in productive struggle but also challenge students to make connections between concepts and to other relevant knowledge. Although there are appropriate times to use low-level cognitive demand tasks, a heavy or sole emphasis on tasks of this type will not lead to a relational understanding of mathematics. As an example of different levels of tasks, consider the degree of reasoning required if you ask students to find the average of five given numbers versus if you ask them to find five numbers whose average is 35. The first task only requires students to find the average of five numbers. The second task requires them to use number sense and their understanding of average to generate five reasonable numbers that will result in a given average. As a consequence of working on this second task, students have potential opportunities to think about and use number relationships while they work on their computational skills for finding averages.

Multiple Entry and Exit Points

A problem or task that has multiple entry points has varying degrees of challenge within it or can be approached in a variety of ways. One of the advantages of a problem-based approach is that it can help accommodate the diversity of learners in every classroom because students are encouraged to use a strategy that makes sense to them instead of using a predetermined strategy that they may or may not be ready to use successfully. Some students may initially use less efficient approaches, such as guess and check or counting, but they will develop more advanced strategies through effective questioning by the teacher and by reflecting on other students' approaches. For example, for the task of finding five numbers whose average is 35, one student may use a guess-and-check approach, using five random numbers to see if their average is 35, while another student may use a more systematic

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Figure 2.1 Levels of cognitive demand.

Low-Level Cognitive Demand Tasks	High-Level Cognitive Demand Tasks
Memorization • Involve producing previously learned facts, rules, formulas, or definitions or memorizing • Are routine, in that they involve exact reproduction of previously learned procedures • Have no connection to related concepts	**Procedures with Connections** • Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas • Suggest general procedures that have close connections to underlying conceptual ideas • Are usually represented in multiple ways (e.g., visuals, manipulatives, symbols, problem situations) • Require that students engage with the conceptual ideas that underlie the procedures in order to successfully complete the task
Procedures without Connections • Use procedures specifically called for • Are straightforward, with little ambiguity about what needs to be done and how to do it • Have no connection to related concepts • Are focused on producing correct answers rather than developing mathematical understanding • Require no explanations or explanations that focus on the procedure only	***Doing* Mathematics** • Require complex and nonalgorithmic thinking (i.e., nonroutine—without a predictable, known approach) • Require students to explore and to understand the nature of mathematical concepts, processes, or relationships • Demand self-monitoring or self-regulation of students' own cognitive processes • Require students to access relevant knowledge in working through the task • Require students to analyze the task and actively examine task constraints • Require considerable cognitive effort

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approach, such as starting with five 35s and then moving part of one 35 to another 35 until he or she has five different numbers. Still another child may reason that if the five numbers were all 35s, then their sum would be 5×35, or 175, so partitioning 175 into five different parts would result in five numbers whose average is 35.

Tasks should also have multiple exit points, or various ways that students can demonstrate an understanding of the learning goals. For example, students might draw a diagram, write an equation, use manipulatives, or act out a problem to demonstrate their understanding.

Consider the opportunities for multiple entry and exit points in the following tasks.

TASK 1:

If there are 1 red tile, 2 yellow tiles, and 1 blue tile in a bag, what is the probability of pulling out a red tile?

TASK 2:

The probability of an event is $\frac{1}{4}$. Describe what the event (situation) might be. Explain how you might use dice, a spinner, or some other tool to simulate the situation.

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Stop and Reflect

To what degree do these tasks offer opportunities for multiple entry and exit points? ■

In the first task, students will gain some experience in thinking about the probability of one event in one situation, but they will miss any opportunity to think deeply about what a probability of $\frac{1}{4}$ means and how it can represent various situations. The second task offers more opportunity for students to engage with the task in a variety of ways, which also offers the teacher more information about each student's level of understanding. For example, do students select examples with more than four objects (e.g., drawing hearts from a deck of cards)? Do students create tree diagrams to help them reason through the situation? Do students create events that are all equally likely, or do they create events that are more likely than others? Can students think about compound events that result in a probability of $\frac{1}{4}$? Clearly, the second task offers many more opportunities for all students to engage in the task in a variety of ways.

Teaching Tip

Before giving a selected task to your class, anticipate several possible responses to the task, including possible misconceptions, and think about how you might address these responses. Anticipating the responses gives you time to consider how you will respond to various approaches, and it also helps you to quickly recognize different strategies and misconceptions when students are working on the task.

Relevant and Well-Designed Contexts

One of the most powerful aspects of teaching through problem solving is that the problem that begins the lesson can get students excited about learning mathematics. Compare the following two sixth-grade introductory tasks on ratios. Which one do you think would be more interesting to students?

> Classroom A: "Today we are going to explore ratios and see how ratios can be used to compare amounts."
>
> Classroom B: "In a minute, I am going to read to you a passage from *Harry Potter* about how big Hagrid is. We are going to use ratios to compare our heights and widths to Hagrid's height and width."

Your goal as a teacher is to design problems that provide specific parameters, constraints, or structure that will support the development of the mathematical ideas you want students to learn. But possibly even more important, familiar and interesting contexts increase students' engagement. In this example, literature was used to engage students. Contexts can also be used to learn about cultures, such as those of the students in your classroom, and can also be used to link to other disciplines (e.g., science, social studies).

Orchestrating Classroom Discourse

Classroom discourse refers to the interactions among all the participants that occur throughout a lesson—in a whole-class setting, in small groups, between pairs of students, and with the teacher. The purpose of discourse is not for students to state their answers and get validation from the teacher but to engage all learners and keep the cognitive demand high (Breyfogle & Williams, 2008–2009; Kilic, Cross, Ersoz, Mewborn, Swanagan, & Kim, 2010; Smith, Hughes, Engle, & Stein, 2009).

Classroom Discussions

The value of student talk throughout a mathematics lesson cannot be overemphasized. As students describe and evaluate solutions to tasks, share approaches, and make conjectures,

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learning will occur in ways that are otherwise unlikely to take place. As they listen to other students' ideas, they come to see the varied approaches in how problems can be solved and see mathematics as something that they can do. Questions such as those that ask students whether they would do it differently next time, which strategy made sense to them (and why), and what caused problems for them (and how they overcame them) are critical in developing mathematically proficient students. Orchestrating discourse after students have worked on problem(s) is particularly important as it is this type of discussion that helps students connect the problem to more general or formal mathematics and make connections to other ideas.

Implementing effective discourse in the classroom can be challenging. Finding ways to encourage students to share their ideas and to engage with others about their ideas is essential to productive discussions. Consider the following research-based recommendations that can be useful in a whole-class setting, in small groups, and in peer-to-peer discussions (Chapin, O'Connor, & Anderson, 2009; Rasmussen, Yackel, & King, 2003; Stephan & Whitenack, 2003; Wood, Williams, & McNeal, 2006; Yackel & Cobb, 1996).

- ***Clarify students' ideas in a variety of ways.*** You can restate students' ideas as questions in order to verify what they did as well as what they meant to confirm what you've heard or observed. You can also apply precise language and make significant ideas more apparent. Paying attention to students' ideas sends the message that their ideas are valued, and therefore this is a key step to encouraging participation of individual students. In addition, modeling how to ask clarifying questions demonstrates to students that it is all right to be unsure and that asking questions is appropriate. It is important to keep in mind that although you may understand a student's ideas and reasoning, there may be students in the class who do not. So look for opportunities to ask clarifying questions even if you do not need clarification. You can also ask students to restate someone else's ideas in their own words in order to ensure that ideas are stated in a variety of ways and to encourage students to listen to one another. This strategy of clarification is important for English language learners (ELLs) because it reinforces language and enhances comprehension.
- ***Emphasize reasoning.*** Ask follow-up questions whether the answer is right or wrong to place an emphasis on the reasoning process. Your role is to understand students' thinking (not to lead them to the correct answer and move on). So, follow up with probes to learn more about their answers and their reasoning. Sometimes, you will find that what you assumed they were thinking is not correct. Also, if you follow up only on wrong answers, students quickly figure this out and get nervous when you ask them to explain their thinking. In addition, move students to more conceptually based explanations when appropriate. For example, if a student says that he knows 4.17 is more than 4.1638, ask him (or another student) to explain why this is so. You can also ask students what they think of the idea proposed by another student, or ask if they see a connection between two classmates' ideas or between a classmate's idea and a concept previously discussed.
- ***Encourage student–student dialogue.*** You want students to think of themselves as capable of making sense of mathematics so that they do not always rely on the teacher to verify the correctness of their ideas. Encouraging student-to-student dialogue can help build this sense of self. Students are also more likely to question one another's ideas than the teacher's ideas. When students have different solutions, ask them to discuss one another's solutions. Or ask someone to rephrase another student's ideas or to add something further to someone else's ideas. Provide opportunities that allow students to share their ideas in small groups or with a peer. This will ensure that all students are able to participate in sharing because not all students will be able to share during every whole-class discussion. Before a whole-class discussion, students can

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Figure 2.2 Examples of teacher prompts for supporting classroom discussions.

Clarify Students' Ideas	"You used a unit ratio to find the price for 15 pounds?" "So, first you recorded your measurements in a table?" "What parts of your drawing relate to the numbers from the story problem?" "Who can share what Ricardo just said, but using your own words?"
Emphasize Reasoning	"Why does it make sense to start with that particular number?" "Explain how you know that your answer is correct." "Can you give an example?" "Do you see a connection between Julio's idea and Rhonda's idea?" "What if ...?" "Do you agree or disagree with Johanna? Why?"
Encourage Student–Student Dialogue	"Who has a question for Vivian?" "Turn to your partner and explain why you agree or disagree with Edwin." "Talk with Yerin about how your strategy relates to hers."

practice their explanations with a peer, which is one way to support ELLs and other students with special needs during mathematical discussions. See Chapters 5 and 6 for other ideas about how to support these particular groups of students with mathematical discussions. Figure 2.2 offers examples of teacher prompts that can support classroom discussions.

Be sure to explain to students that after they hear a question or a prompt they will have time to think so that silence in the classroom does not feel uncomfortable. For example, you can say, "This question is important. Let's take some time to think about it." There will be times when no one responds to your question or prompt. If the situation gets awkward, make sure students understand the question or prompt, then ask them to talk with a partner and try the discussion again.

How Much to Tell and Not to Tell

When teachers teach mathematics *through* problem solving, one of the most perplexing dilemmas is how much, if anything, to tell. On one hand, telling can diminish what is learned and lower the level of challenge in a lesson. On the other hand, telling too little can sometimes leave students floundering, or not productively struggling. Following are suggestions about three things that you need to tell students:

- *Introduce mathematical conventions.* Symbols, such as $\surd$ and x^3, and notations, such as $(1, ^-2)$, are conventions. Terminology is also a convention. As a rule of thumb, symbolism and terminology should be introduced *after* concepts have been developed and then specifically as a means of expressing or labeling ideas.
- *Discuss alternative methods.* If an important strategy does not emerge naturally from students, then you should propose the strategy, being careful to identify it as "another" way, not the only or the preferred way.
- *Clarify students' methods and make connections.* You should help students clarify or interpret their ideas and point out related concepts. A student may divide $\frac{5}{6}$ by $\frac{1}{3}$ by

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thinking about how many one-thirds can be measured out or subtracted from $\frac{5}{6}$. This strategy can be related to measurement division with whole numbers, such as thinking of $12 \div 3$ as how many 3s can be measured out or subtracted from 12. Drawing everyone's attention to this connection can help other students see the connection while also building the confidence of the student who originally proposed the strategy (Hiebert et al., 1997).

Representations: Tools for Problem Solving, Reasoning, and Communication

A representation can be thought of as a kind of tool, such as a diagram, graph, symbol, or manipulative, that expresses a mathematical idea or concept. Representations are not ends in themselves to be learned for the sake of learning but are valuable tools in problem solving, reasoning, and communicating about mathematical ideas. Representations can help you think through a problem and better communicate your ideas to another person. How you represent the ideas in the problem will likely influence your solution process. In fact, the representations that students choose to use can provide valuable insight into their ways of interpreting and thinking about the mathematical ideas at hand.

Models or representations give learners something with which they can explore, reason, and communicate as they engage in problem-based tasks. The goal of using representations is so that students can manipulate ideas, not manipulate symbols in a rote manner. By using personally meaningful representations to manipulate and communicate about mathematical ideas, students will make connections among mathematical ideas (relational understanding) and move toward mathematical proficiency.

Tips for Using Representations in the Classroom

Because different representations can illuminate different aspects of a mathematical idea, multiple representations should be explored and encouraged. The more ways that students are given to think about and test an emerging idea, the better they will correctly form and integrate it into a rich web of concepts and thereby develop a relational understanding. Figure 2.3 illustrates various representations for demonstrating an understanding of any topic. Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Clement, 2004; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003; NCTM, 2000). Strengthening the ability to move between and among representations improves students' understanding and retention of ideas.

The following are rules of thumb for using representations or models in the classroom:

- Introduce new representations or tools by showing how they can represent the *ideas* for which they are intended. But keep in mind that because the representations are not the concepts, some students may not "see" what you see.
- Allow students (in most instances) to select freely from available tools to use in solving problems.
- Encourage students to create their own representations. Look for opportunities to connect these student-created representations to more conventional representations.
- Encourage the use of a particular representation when you believe it would be helpful to a student having difficulty.

Teaching Tip

Pay attention to students' choices of representations, and use those representations as starting points for dialogues with them about their thinking. What they find important may be surprising and informative at the same time.

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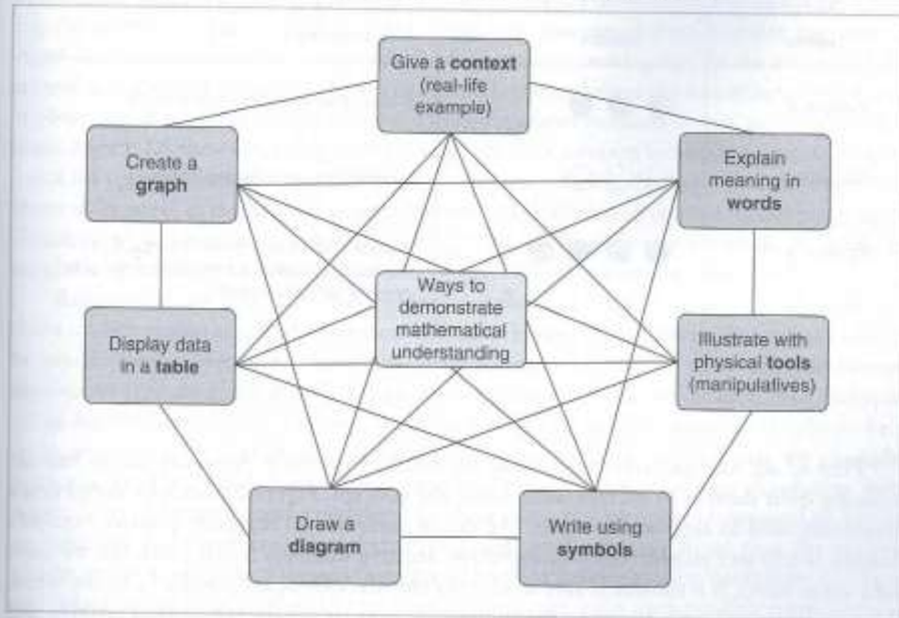
Figure 2.3

Mathematical understanding can be demonstrated through these different representations of mathematical ideas. Translations between each can help students develop new concepts and demonstrate a richer understanding.



- Ask students to use representations, such as diagrams and manipulatives, when they explain their thinking. This will help you gather information about students' understanding of the idea and also their understanding of the representations that have been used in the classroom. It can also be helpful to other students in the classroom who may be struggling with the idea or the explanation being offered.
- In creating tasks and when facilitating classroom discussions, focus on making connections among the different representations used (and make sure each is understood). Helping students make these connections is very important to their learning.

Note that problems can start with one representation (e.g., a story problem) and ask the student to translate the information to another representation (e.g., an equation); yet a student might get to the final representation by working through other representations (e.g., creating a table or drawing a picture to get to an equation). These representations, whether student-created or more conventional, are critical in supporting students' reasoning and actually progressing toward more abstract symbolic representations.

Manipulatives

Let's turn to one kind of representation that is commonly used to support students' learning of mathematics—manipulatives, or concrete objects. Any time a concept is new, regardless of the ages of the students, manipulatives can be a positive factor in students' learning. However, just using manipulatives, particularly in a rote manner, does not ensure that students will understand. It is important to consider how manipulatives can help, or fail to help, students construct mathematical knowledge.

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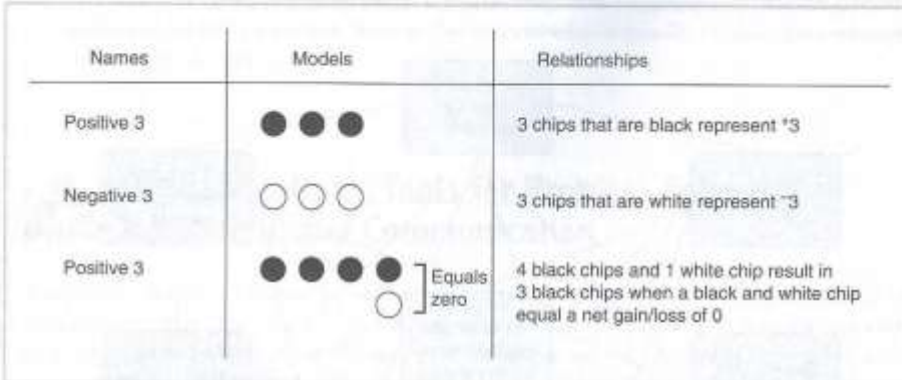
Figure 2.4

Objects and names of objects are not the same as mathematical ideas and relationships between objects.



First of all, manipulatives alone have no inherent meaning. A person has to impose meaning onto them. The manipulative is not the concept. Figure 2.4 shows colored chips commonly used to represent integers. We define one color to represent positive numbers (usually black) and another color to represent negative numbers (usually red, but we have used white here). If a student is able to identify the black chips as "positive" and the white chips as "negative," does this mean the student has constructed the concepts of positive and negative numbers and can operate with them in meaningful ways? No, all you know for sure is that the student has learned the names typically assigned to the manipulatives. In fact, there is evidence across a range of grades that students struggle with negative numbers (Mukhopadhyay, 1997; Vlassis, 2004). Calculating with integers can become a lesson in memorization when students are rushed to follow rules such as "two like signs become a positive" and "two unlike signs become a negative." Consequently, teachers attempt to support students' work by using manipulatives such as colored chips. However, the concept of "negative" must be created by students in their own minds and imposed on the manipulative used to represent the concept (connecting money to the chips using a profit/loss context can help students construct this meaning). Through discussions that explicitly focus on the mathematical concepts over time, the connections between manipulatives and related concepts are developed.

Teaching Tip

It is incorrect to say that a manipulative or object "illustrates" or shows a concept. Manipulatives can help students visualize relationships and talk about them, but what they see are the manipulatives, not concepts.

Second, the most widespread misuse of manipulatives occurs when teachers tell students, "Do exactly as I do." There is a natural temptation to get out the materials and show students exactly *how to use them*. Students mimic the teacher's directions, and it may even look as if they understand, but they may just be following what they see. A rote procedure with a manipulative is still just that—a rote procedure.

A third and related misuse of manipulatives occurs when teachers always tell students which manipulative to use for a given problem. Students need opportunities to choose their own representations to use when reasoning through a problem (Mathematical Practice 5: Use appropriate tools strategically) and when communicating their ideas to others.

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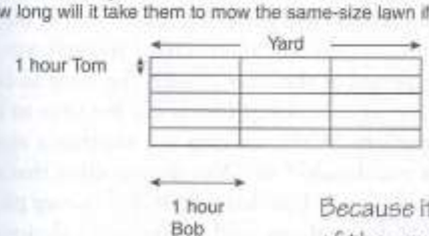
Visuals and Other Tools

There are other ways for students to represent and illustrate mathematical concepts. Drawings are one option and are important for a number of reasons. First, when students draw, you learn more about what they do or do not understand. For example, if students are showing the part–part ratio 2:3 with their own drawings, you can observe whether they understand that the whole has five parts, with the first quantity making up $\frac{2}{5}$ of the whole and the second quantity making up $\frac{3}{5}$ of the whole. Second, manipulatives can sometimes restrict how students can model a problem, whereas a drawing allows students to use any strategy they want. Figure 2.5 shows an example of a seventh grader's solution for solving a ratio problem. Look for opportunities to use students' representations during classroom discussions to help them make sense of the more abstract mathematical symbols and computational procedures. Third, when students create a drawing you tend to get different representations, providing an excellent opportunity to compare and contrast the various approaches and visuals.

Representations generated and manipulated through technology can also support students as they reason about and communicate their mathematical ideas. Changes can usually be made to situations more quickly by using a computer than by using physical manipulatives or student-generated drawings, leaving more time for exploration. For example, using interactive graphing software, students can quickly make changes to graphs to help them analyze and interpret how situations change as variables change (see, for example, www.nctm.org/standards/content.aspx?id=25092). As another example, simulation software allows students to perform several trials in just a few seconds, as opposed to having to complete the actual experiment multiple times, again allowing more time for analysis and interpretation of the situation (e.g., go to http://nlvm.usu.edu/en/nav/grade_g_3.html and select "Coin Toss"). Plus, virtual manipulatives can help students link manipulatives to symbols. For example, some websites display decimals with numerals and computerized base-ten blocks, and as changes are made to the base-ten blocks, students can see the results

Figure 2.5 A seventh grader shows her thinking about a ratio problem.

Bob can mow a yard in 3 hours. Tom can mow the same-size yard in 5 hours. How long will it take them to mow the same-size lawn if they do it together?



Because it takes Bob an hour to mow $\frac{5}{15}$ of the yard and Tom an hour to mow $\frac{3}{15}$ of the yard, together they can mow $\frac{8}{15}$ of the yard in an hour. Or they can mow 8 parts of the lawn in an hour. After an hour they still have 7 parts to mow. So it will take them $\frac{7}{8}$ of an hour to mow the rest of the yard. Or it takes them $1\frac{7}{8}$ or $\frac{15}{8}$ hours to mow the lawn together.

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of the actions they take on the numeral representation of the decimal (e.g., go to http://nlvm.usu.edu/en/nav/grade_g_3.html and click on "Base Blocks—Decimals"). The dynamic link between these two representations helps students make sense of their activity as well as the numbers. An added bonus with technology is that sometimes the language displayed on the computer program can be changed for ELLs.

Meaningful contexts help students make sense of mathematical ideas. Using real objects, pictures, drawings, and virtual manipulatives can help students relate to and better understand a context, especially one that is unfamiliar. This is particularly important in supporting ELLs or students with disabilities.

A Three-Phase Lesson Format

A three-phase lesson format (*Before, During, After*) provides a structure for teaching mathematics through problem solving (Table 2.1). *Before* refers to the time before students start work on the problem; *During* refers to the time during which students work on the problem; and *After* refers to the discussion that takes place after students work on the problem. The lesson may take one or more class sessions, but the three-phase structure can also be applied to shorter tasks, resulting in a 10- to 20-minute minilesson.

Before

In the *Before* phase of the lesson, you are preparing students to work on the problem. As you plan for the *Before* part of the lesson, analyze the problem you will give to students in order to anticipate students' approaches and possible misinterpretations or misconceptions (Wallace, 2007). This can inform the questions you ask in the *Before* phase of the lesson to clarify students' understanding of the problem (i.e., knowing what it means rather than how they will solve it).

During

In the *During* phase of the lesson, students explore the problem (alone, with partners, or in small groups). This is one of two opportunities you will get in the lesson to find out what your students know, how they think, and how they are approaching the task you have given them (the other is in the discussion period of the *After* phase). You want to convey a genuine interest in what students are doing and thinking. This is not the time to evaluate or to tell students how to solve the problem. When students ask whether a result or method is correct, ask students, "How can you decide?" or "Why do you think that might be right?" or "How can we tell if that makes sense?" Use this time in the *During* phase to identify different representations and strategies students used, interesting solutions, and any misconceptions that arise that you will highlight and address in the *After* phase of the lesson.

After

In the *After* phase of the lesson your students will work as a community of learners, discussing, justifying, and challenging various solutions to the problem that all have just worked on. It is critical to plan for and save ample time for this part of the lesson. Twenty minutes is not at all unreasonable for a good class discussion and sharing of ideas. It is not necessary

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Table 2.1 Teaching Mathematics through Problem Solving Lends Itself to a Three-Phase Structure for Lessons

Lesson Phase	Teacher Actions in a Teaching Mathematics through Problem-Solving Lesson	
Before	Activate prior knowledge.	Begin with a simple version of the task; connect to students' experiences; brainstorm approaches or solution strategies; estimate or predict whether tasks involve a single computation or are aimed at the development of a computational procedure.
	Be sure the problem is understood.	Have students explain to you what the problem is asking. Go over vocabulary that may be troubling. Caution: This does *not* mean that you are explaining how to do the problem—just that students should understand what the problem is about.
	Establish clear expectations.	Tell students whether they will work individually, in pairs, or small groups, or if they will have a choice. Tell them how they will share their solutions and reasoning.
During	Let go!	Although it is tempting to want to step in and "help," hold back and enjoy observing and learning from students.
	Notice students' mathematical thinking.	Base your questions on students' work and their responses to you. Use questions like "Tell me what you are doing"; "I see you have started to [multiply] these numbers. Can you tell me why you are [multiplying]?" [substitute any process/strategy]; "Can you tell me more about...?"; "Why did you...?"; "How does your diagram connect to the problem?"
	Provide appropriate support.	Look for ways to support students' thinking and avoid telling them how to solve the problem: Ensure that students understand the problem (e.g., "What do you know about the problem?"); ask the student what he or she has already tried (e.g., "Where did you get stuck?"); suggest that the student use a different strategy (e.g., "Can you draw a diagram?"; "What if you used cubes to act out this problem?"; "Is this like another problem we have solved?"); create a parallel problem with simpler values (Jacobs & Ambrose, 2008).
	Provide worthwhile extensions.	Challenge early finishers in some manner that is related to the problem just solved. Possible questions to ask are "I see you found one way to do this. Are there any other solutions? Are any of the solutions different or more interesting than others?" Some good questions for extending thinking are, "What if...?" or "Would that same idea work for...?"
After	Promote a community of learners.	You must teach students about your expectations for this time and how to interact respectfully with their peers. Role-play appropriate (and inappropriate) ways of responding to each other. The "Orchestrating Discourse" section provides strategies and recommendations for how to facilitate discussions that help create a community of learners.
	Listen actively without evaluation.	The goal here is noticing students' mathematical thinking and making that thinking visible to other students. Avoid judging the correctness of an answer so that students are more willing to share their ideas. Support students' thinking without evaluation by simply asking what others think about a student's response.
	Summarize main ideas and identify future problems.	Formalize the main ideas of the lesson, helping to highlight connections among strategies or different mathematical ideas. In addition, this is the time to reinforce appropriate terminology, definitions, and symbols. You may also want to lay the groundwork for future tasks and activities.

to wait for every student to finish. Here is where much of the learning will occur as students reflect individually and collectively on the ideas they have explored. This is the time to reinforce precise terminology, definitions, or symbols. After students have shared their ideas, formalize the main ideas of the lesson, highlighting connections among strategies or different mathematical ideas.

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What Do I Do When a Task Doesn't Work?

Sometimes students may not know what to do with a problem you pose, no matter how many hints and suggestions you offer. Do not give in to the temptation to "tell them." But when you sense that a task is not moving forward, don't spend days just hoping that something wonderful may happen. You may need to regroup and offer students a simpler but related task that gets them prepared for the one that proved too difficult. If that does not work, set it aside for the moment. Ask yourself why it didn't work well. Did the students have the prior knowledge they needed? Was the task too advanced? Consider what might be a way to step back or step forward in the content in order to support and challenge students. Nonetheless, trust that teaching mathematics *through* problem solving offers students the productive struggle that will allow them to develop understanding and become mathematically proficient.

Stop and Reflect

Describe what is (and isn't) meant by "teaching mathematics *through* problem solving." What do you foresee to be some opportunities and challenges to implementing problem-based mathematics tasks effectively in your classroom? ■

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